Math 451: Introduction to General Topology
Lecture 14

The profinite topology on $\mathbb{Z}$ (continued).

Claim. The profinite topology on $\mathbb{Z}$ is metrizable, e.g. by the metric
$$d(x,y) := \|x-y\|_2,$$
where $\|z\|_2 := \sum_{\substack{n \geq 1 \\ n \nmid z}} \frac{1}{2^n}$.

Proof. HW.

We use this topology to give a funny proof (by Furstenberg) that $\exists$ ∞-many primes. First let's recall the usual proof, which hinges on the fact that every integer $z \in \mathbb{Z}$, $z \neq \pm 1$, is divisible by a prime number, i.e. $\exists$ prime p such that $\exists k \in \mathbb{Z}$ with $z = kp$.

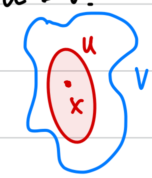
Prop. $\exists$ ∞-many primes in $\mathbb{Z}$.

Proof 1. Suppose otherwise, so $\exists$ only finitely many primes $p_1, p_2, \dots, p_n$. Then $z := p_1 p_2 \cdots p_n + 1$ is not divisible by any prime p_i since if $z = kp_i$ for some $k \in \mathbb{Z}$, then $kp_i = p_1 p_2 \cdots p_n + 1$, so $1 = kp_i - p_1 p_2 \cdots p_n = p_i(k - m)$ is divisible by p_i, a contradiction. But z not being divisible by any prime contradicts the above statement. $\square$

Proof 2 (Furstenberg). Note that $z \in \mathbb{Z}$ is divisible by a prime $p \iff z \in p\mathbb{Z}$, which implies
$$\mathbb{Z} \setminus \{\pm 1\} = \bigcup_{p \text{ prime}} p\mathbb{Z}$$
because every $z \in \mathbb{Z} \setminus \{\pm 1\}$ is divisible by a prime. If there were only finitely many primes $p_1, p_2, \dots, p_n$, we would have
$$\mathbb{Z} \setminus \{\pm 1\} = \bigcup_{i=1}^{n} p_i \mathbb{Z},$$

so $\mathbb{Z}\setminus\{\pm 1\}$ is clopen since it is a finite union of clopen sets. In particular, $\{\pm 1\}$ is open, which contradicts that open sets in the profinite topology are infinite (since they are unions of arithmetic progressions bcse the latter form a basis).

Def. Let X be a topological space and $x \in X$. A **neighbourhood basis at x** is a collection $\mathcal{B}_x$ of open sets containing x such that for every open $V \ni x$ $\exists\, U \in \mathcal{B}_x$ with $U \subseteq V$.



Examples. (a) In any top. space $(X, \mathcal{T})$ and $x \in X$, the set $\mathcal{B}_x := \{U \in \mathcal{T} : U \ni x\}$ is a neighbourhood basis at x.

(b) In a metric space X, for each $x \in X$, the collection of all open balls centered at x is a neighbourhood basis at x. In fact, the collection $\{B_{\frac{1}{n}}(x) : n \in \mathbb{N}^+\}$ is a neighbourhood basis and it is ctbl.

(c) In any top. space X, if $\mathcal{B}$ is a basis for the topology, then

$$\mathcal{B}_x := \{U \in \mathcal{B} : U \ni x\}$$

is a neighbourhood basis at x.

Proof. If $V \ni x$ is open, then V is a union of sets in $\mathcal{B}$, hence $x \in U$ for some $U \in \mathcal{B}$, so $U \in \mathcal{B}_x$. $\square$

Prop (converse to example (c)). In a top. space X, if $\mathcal{B}_x$ is a neighbourhood basis at x, for each $x \in X$, then $\mathcal{B} := \bigcup_{x \in X} \mathcal{B}_x$ is a basis for the topology.

Proof. Let $V \subseteq X$ be open. Then for each $x \in V$, $\exists\, U_x \in \mathcal{B}_x$ with $x \in U_x \subseteq V$. But then

$$V = \bigcup_{x \in V} U_x.$$

$\square$

Subspace topology (aka relative topology).

Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. We can make Y into a topological space as follows: let $\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\}$.

(aka relative)

Prop. $\mathcal{T}_Y$ is a topology on Y, called the subspace topology of Y.

Proof. $\emptyset = \emptyset \cap Y$ and $Y = X \cap Y$ are both in $\mathcal{T}_Y$. If $\{V_i\}_{i\in I}$ is an arbitrary collection of sets in $\mathcal{T}_Y$, then each $V_i = U_i \cap Y$ for some open $U_i \in \mathcal{T}$, hence

$$\bigcup_{i\in I} V_i = \bigcup_{i\in I} (U_i \cap Y) = \Big(\bigcup_{i\in I} U_i\Big) \cap Y \in \mathcal{T}_Y$$

because $\bigcup_{i\in I} U_i \in \mathcal{T}$. Similarly, one verifies the closedness of $\mathcal{T}_Y$ under finite intersections. □

Examples. (a) If (X,d) is a metric space and $Y \subseteq X$, then the subspace top. on Y is induced by the restriction $d|_Y$ of the metric, i.e. $(Y, d|_Y)$ is a metric space whose open sets are exactly the sets of the form $U \cap Y$, where U is open in X.

(b) Let X be any top. space and $Y \subseteq X$ an open subset. Then the subspace topology on Y consists exactly of those open sets in X which happen to be subsets of Y.
This is because for open $U \subseteq X$, $U \cap Y$ is still open in X.

(c) In the subspace top of $[0,1]$ induced from $\mathbb{R}$, the set $[0,\frac{1}{2})$ is open because $[0,\frac{1}{2}) = (-\frac{1}{2},\frac{1}{2}) \cap [0,1]$.

(d) The $\frac{1}{3}$-Cantor set $C \subseteq [0,1]$ with the subspace top induced by $\mathbb{R}$ is homeomorphic to the Cantor space $2^{\mathbb{N}}$, as proved earlier.

Remark. The properties of a top. space X are not always inherited by its subspaces.

Example. $\mathbb{Q}$, the Cantor set C are both 0-dimensional, while $\mathbb{R}$ is not, in fact the only clopen subsets of $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$ (i.e. $\mathbb{R}$ is connected).

Topological countability.

Def. A topological space X is called:

- separable if there is a ctbl dense set $D \subseteq X$.
- 1^{st} ctbl if there is a ctbl neighbourhood basis at each point $x \in X$.
- 2^{nd} ctbl if there is a ctbl basis.

- **Lindelöf** if every open cover of X admits a ctbl subcover.
  - An open cover of X is a collection $\mathcal{U}$ of open sets such that $X = \bigcup_{U \in \mathcal{U}} U$.

    Example: $\mathcal{U} := \{(n, n+2) : n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$.
  - A subcover of $\mathcal{U}$ is just a subcollection $\mathcal{U}' \subseteq \mathcal{U}$ which is still a cover of X.

Examples. (a) $\mathbb{R}$, more generally $\mathbb{R}^d$ are separable because $\mathbb{Q}^d$ is dense and ctbl.
Also, $\mathbb{R}^d$ is 2^{nd} ctbl because the collection of balls of radii $\frac{1}{n}$, $n \in \mathbb{N}^+$, centered at $\mathbb{Q}^d$ is a basis and is ctbl.

(b) For a ctbl $\Sigma \neq \emptyset$, $\Sigma^{\mathbb{N}}$ is separable because fixing $0 \in \Sigma^{\mathbb{N}}$, the set

$$Q := \{w0^\infty : w \in \Sigma^{<\infty}\}$$

is dense in $\Sigma^{\mathbb{N}}$ because every cylinder $[w]$ has a representative $w0^\infty$, where

$$0^\infty := 0000\ldots,$$

so if $w := w_0 \ldots w_{n-1}$, then $w0^\infty = w_0 \ldots w_{n-1} 0000\ldots$.
Since $\Sigma^{<\infty}$ is ctbl, Q is ctbl.
Σ^∞ is in fact 2^{nd} ctbl since the cylinders form a basis and there are only $\Sigma^{<\infty}$-many cylinders, hence ctbly many.

(c) Profinite topology on $\mathbb{Z}$ is 2^{nd} ctbl because there are only ctbly many arithmetic progressions $a + b\mathbb{Z}$, $a, b \in \mathbb{Z}$, because $\mathbb{Z}$ is ctbl.

Prop. 2^{nd} ctbl $\Rightarrow$ 1^{st} ctbl + separable.

Proof. Let $\mathcal{B}$ be a ctbl basis for the top. of X. Then for each $x \in X$,

$$\mathcal{B}_x := \{U \in \mathcal{B} : U \ni x\}$$

is a neighbourhood basis at x and is ctbl because $\mathcal{B}_x \subseteq \mathcal{B}$.

For separability, pick (using AC) one point from each set in $\mathcal{B}$, i.e. for each $U \in \mathcal{B}$, let $x_U \in U$, then take $D := \{x_U : U \in \mathcal{B}\}$. By definition, D meets every set in $\mathcal{B}$, hence every open set since every open set is a union of sets in $\mathcal{B}$. $\square$

The converse of this proposition fails and we will start with a counter-example next time.